ON INVARIANTS OF A SET OF ELEMENTS OF A SEMISIMPLE LIE ALGEBRA

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Abstract. Let G be a complex reductive algebraic group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{h}$ a reductive subalgebra of $\mathfrak{g}$, n a positive integer. Consider the diagonal actions $G:\mathfrak{g}^n, N_G(\mathfrak{h}):\mathfrak{h}^n$. We study a relation between the algebra $\mathbb{C}[\mathfrak{h}^n]^{N_G(\mathfrak{h})}$ and its subalgebra consisting of restrictions to $\mathfrak{h}^n$ of elements of $\mathbb{C}[\mathfrak{g}^n]^G$.

Contents

1. Introduction

Let G be a reductive algebraic group over the field $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a reductive algebraic subalgebra of $\mathfrak{g}$ and $\widetilde{H}=N_G(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in the group G.

Let n be a positive integer. One has the diagonal actions $G:\mathfrak{g}^n,\widetilde{H}:\mathfrak{h}^n$. Consider a subalgebra of $\mathbb{C}[\mathfrak{h}^n]$ whose elements are restrictions of elements of $\mathbb{C}[\mathfrak{g}^n]^G$ to $\mathfrak{h}^n$. We denote this algebra by $\mathbb{C}[\mathfrak{h}^n]^G$. Clearly, $\mathbb{C}[\mathfrak{h}^n]^G\subset\mathbb{C}[\mathfrak{h}^n]^{\widetilde{H}}$. It is interesting to ask how different these two algebras can be.

It is more convenient to translate this question into geometric language. As usual, we denote by $\mathfrak{h}^n/\!/\widetilde{H}$ the categorical quotient for the action $\widetilde{H}:\mathfrak{h}^n$. In other words, $\mathfrak{h}^n/\!/\widetilde{H}=\mathrm{Spec}(\mathbb{C}[\mathfrak{h}^n]^{\widetilde{H}})$. Put $\mathfrak{h}^n/\!/G=\mathrm{Spec}(\mathbb{C}[\mathfrak{h}^n]^G)$. The inclusion $\mathbb{C}[\mathfrak{h}^n]^G\hookrightarrow\mathbb{C}[\mathfrak{h}^n]^{\widetilde{H}}$ induces the morphism of algebraic varieties $\psi_n:\mathfrak{h}^n/\!/H\rightarrow\mathfrak{h}^n/\!/G$. The aim of this paper is to answer the following questions: is ψ_n isomorphism, birational, bijective, finite morphism?

The property of ψ_n being bijective has a nice alternative description. Namely, $\psi_n, n\geqslant 2$, is bijective iff for any reductive algebraic group F and any homomorphisms $P_1,P_2:F\rightarrow H$ the following conditions are equivalent:

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(1) There is $g \in G$ such that $\mathrm{Ad}(g) \circ \rho_1 = \rho_2$.
(2) There is $g \in \widetilde{H}$ such that $\mathrm{Ad}(g) \circ \rho_1 = \rho_2$.

Here $\rho_i = dP_i : \mathfrak{f} \to \mathfrak{h}, i = 1, 2$, is the corresponding homomorphism of Lie algebras.

The starting point for our work is E.B. Vinberg's paper [Vi], where the morphism $\Psi_n : H^n//\widetilde{H} \to H^n//G$ defined analogously to ψ_n was studied (here H is a reductive subgroup of G). The main result of that paper is that Ψ_n is the morphism of normalization for $n \geqslant 2$.

Now we list our main results.

At first, the morphism ψ_n is always finite (Proposition 2.5). If $n > 1$ it is also birational (Proposition 2.4). Therefore, ψ_n is the morphism of normalization for $n > 1$. In general, ψ_1 is not birational. However, the following statement holds

Theorem 1.1. *Suppose that $G = \mathrm{GL}_n$ and $\mathfrak{h}$ is a simple algebra different from $\mathfrak{so}_9, \mathfrak{sp}_8, \mathfrak{so}_{16}, \mathfrak{sl}_8, \mathfrak{sl}_9$. Then ψ_1 is birational. If $\mathfrak{h}$ is one of five algebras listed above, then for some positive integer m there exists an embedding $\mathfrak{h} \hookrightarrow \mathfrak{gl}_m$ such that the corresponding morphism ψ_1 is not birational.*

If ψ_n is bijective, then ψ_1 is also bijective. For some G and $\mathfrak{h}$ the converse is true. To describe such pairs we need the following definition:

Definition 1.2. Let H be a reductive algebraic group and $\mathfrak{h}$ be its Lie algebra. Suppose that $\mathfrak{h}$ does not contain simple ideals isomorphic to E_6, E_7, E_8. Further, suppose that for every ideal $\mathfrak{h}_1 \subset \mathfrak{h}$ isomorphic to $\mathfrak{so}_{2k}, k > 3$, there exists $h \in H$ such that the restriction of $\mathrm{Ad}(h)$ to a simple ideal $\mathfrak{h}_2 \subset \mathfrak{h}$ is an outer involutory automorphism of $\mathfrak{h}_2$ if $\mathfrak{h}_2 = \mathfrak{h}_1$ and the identity otherwise (the claim of $\mathrm{Ad}(h)|_{\mathfrak{h}_1}$ being involution is essential only for $\mathfrak{h}_1 \cong \mathfrak{so}_8$). Then we say that H is a *group of type I*.

Put $\overline{H} = \widetilde{H}/Z_G(\mathfrak{h})$.

Theorem 1.3. *If $\overline{H}$ is a group of type I and ψ_1 is bijective, then so is ψ_n.*

However, in some cases ψ_2 is not bijective.

Proposition 1.4. *If G is a group of type I, $\mathfrak{h}$ is a simple algebra and $\overline{H}$ is not a group of type I, then ψ_n is not bijective for $n > 1$.*

In fact, the claim of $\mathfrak{h}$ being simple can be omitted, but we do not prove it.

Suppose now that $G = \mathrm{GL}_m$. It is known from classical invariant theory that the algebra $\mathbb{C}[\mathfrak{g}^n]^G$ is generated by polynomials of the form $f(X_1, \ldots, X_n) = \mathrm{tr}(X_{i_1} X_{i_2} \ldots X_{i_k})$. Clearly, if ψ_n is isomorphism, then ψ_1 is isomorphism too. In some cases the opposite is true.

Proposition 1.5. *The algebra $\mathbb{C}[\mathfrak{h}^n]^G$ is generated by elements of the form $f(L(X_1, \ldots, X_n))$, where $f \in \mathbb{C}[\mathfrak{h}]^G$ and L is a Lie polynomial on $X_1, \ldots, X_n$.*

Using Proposition 1.5 one can prove that ψ_n is isomorphism if so is ψ_1 for $\mathfrak{h} = \mathfrak{sp}_{2m}$, $\mathfrak{h} = \mathfrak{so}_{2m+1}$, $\mathfrak{h} = \mathfrak{so}_{2m}$ with $\overline{H} \cong \mathrm{Ad}(\varnothing_{2m})$, $\mathfrak{h} = \mathfrak{sl}_m$ with $\overline{H} = \mathrm{Ad}(\mathrm{SL}_m)$.

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2. Finiteness and birationality of ψ_n in general case

Let $G, \mathfrak{g}, \mathfrak{h}, \widetilde{H}$ be as above and n be a positive integer. The natural map $\Phi_n : \mathfrak{h}^n \to \mathfrak{g}^n//G$ is constant on $\widetilde{H}$-orbits. Therefore there is a natural morphism $\varphi_n : \mathfrak{h}^n//\widetilde{H} \to \mathfrak{g}^n//G$. The closure of $\varphi_n(\mathfrak{h}^n//H)$ coincides with $\mathfrak{h}^n//G$.

The following proposition is due to Richardson [Ri].

Proposition 2.1. *Let G be a reductive group, $\mathfrak{g}$ its Lie algebra and n a positive integer. Consider the action $G:\mathfrak{g}^n$ as above. The orbit of an n-tuple $\mathbf{x}=(x_1,\ldots,x_n)\in\mathfrak{g}^n$ is closed (resp. contains 0 in its closure) iff the algebraic subalgebra of $\mathfrak{g}$ generated by $x_1,\ldots,x_n$ is reductive (resp. consists of nilpotent elements).*

Corollary 2.2. *$\mathfrak{h}^n//\widetilde{H}$ (respectively, $\mathfrak{h}^n//G$) can be identified with a set of equivalence classes of n-tuples $(x_1,\ldots,x_n)\in\mathfrak{h}^n$ generating a reductive subalgebra of $\mathfrak{h}$ modulo $\widetilde{H}$- (respectively, G-) conjugacy.*

Lemma 2.3. *Any reductive algebraic Lie algebra $\mathfrak{h}$ can be generated by two elements (as an algebraic algebra). If $\mathfrak{h}$ is commutative, then it can be generated by one element.*

Proof. The proof is completely analogous to the proof of Proposition 2 in [Vi]. □

Proposition 2.4. *Suppose $n>1$. Then ψ_n is birational.*

Proof. The proof is completely analogous to the proof of Theorem 2 in [Vi]. □

Proposition 2.5. *The morphism ψ_n is finite.*

Proof. Denote by $\mathcal{N}_G$ (resp. $\mathcal{N}_{\widetilde{H}}$) the null-cone for the action $G:\mathfrak{g}^n$ (resp. $\widetilde{H}:\mathfrak{h}^n$). It follows from Proposition 2.1, that $\mathcal{N}_G\cap\mathfrak{h}^n=\mathcal{N}_{\widetilde{H}}$. Now our statement follows from one version of Nullstellensatz (see [Kr], ch.2, s. 4.3, Theorem 8). □

Corollary 2.6. *Suppose $n>1$. Then ψ_n is the morphism of normalization.*

Proof. Since $\mathfrak{h}^n//\widetilde{H}$ is normal, this is a direct consequence of Propositions 2.4,2.5. □

3. Birationality of ψ_1 for $G=\mathrm{GL}_n$ and simple algebra $\mathfrak{h}$

In this section we prove Theorem 1.1. Here we assume that $\mathfrak{h}$ is a simple Lie algebra of rank r. Denote by $\alpha_1,\ldots,\alpha_r$ simple roots of $\mathfrak{h}$, by $\pi_1,\ldots,\pi_r$ the corresponding fundamental weights and by P and Q the weight and the root lattices of $\mathfrak{h}$, respectively.

Lemma 3.1. *Let Δ be an irreducible root system in a real vector space V, W its Weyl group, Q the lattice generated by Δ. Denote by $(\cdot,\cdot)$ W-invariant scalar product on V. Let $g\in\mathrm{Aut}(Q)$ be an orthogonal linear operator. If $\Delta\neq C_4$, then $g\in\mathrm{Aut}(\Delta)$.*

Proof. It follows from the construction of root systems (see, for example, [Bou], chapter 6) that if $\Delta\neq C_l$, then elements of Q lying in Δ are all elements satisfying some conditions on their length. For $\Delta=C_l$ the same is true for roots of minimal length. Suppose now that $\Delta=C_l$ and $l\neq 4$. Let g be an element of $\mathrm{O}(V)$ such that the set of elements of Δ of minimal length is invariant under g. Then $g\in W$ and we are done. □

Proof of Theorem 1.1. Denote by $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{h}$. The points of $\mathfrak{h}//\widetilde{H}\cong\mathfrak{h}//\overline{H}$ (respectively, $\mathfrak{h}//G$) are in one-to-one correspondence with equivalence classes of semisimple elements of $\mathfrak{h}$ modulo $\overline{H}$- (respectively G-) conjugacy (see Corollary 2.2). If t is an element of $\mathfrak{t}$ in general position, then $gt\in\mathfrak{t}$ for some $g\in G$ implies $g\in N_G(\mathfrak{t})$. Thus, ψ_1 is birational iff $N_G(\mathfrak{t})/Z_G(\mathfrak{t})=N_{\overline{H}}(\mathfrak{t})/Z_{\overline{H}}(\mathfrak{t})$.

Suppose that $\mathfrak{h}\neq\mathfrak{sp}_8,\mathfrak{so}_9,\mathfrak{sl}_8,\mathfrak{sl}_9,\mathfrak{so}_{16}$. Denote by φ an embedding of $\mathfrak{h}$ into $\mathfrak{gl}_m$. We identify $\mathfrak{h}$ and $\varphi(\mathfrak{h})$. Denote by N the image under the natural homomorphism of $N_G(\mathfrak{t})$ in $\mathrm{GL}(\mathfrak{t})$. It is clear that N contains the Weyl group of $\mathfrak{h}$. Now we prove that $N\subset\mathrm{Aut}(\Delta)$.

For $x,y\in\mathfrak{gl}_m$ put $(x,y)=\mathrm{tr}(xy)$. The restriction of $(\cdot,\cdot)$ to $\mathfrak{t}$ is an N-invariant non-degenerate symmetric bilinear form. Its restriction to $\mathfrak{t}(\mathbb{R})$ is a scalar product. Further, we notice that the lattice X generated by the weights of φ is invariant under the action of N.

Obviously, $Q\subset X\subset P$. It follows from Lemma 3.1, that if $X=Q$ and $N\not\subset\mathrm{Aut}(\Delta)$, then $\Delta=C_4$. Suppose now that $X=P$. Then the dual root lattice $Q^\vee$ is invariant under the action of N on $\mathfrak{t}$. Lemma 3.1 implies that if $N\not\subset\mathrm{Aut}(\Delta)$, then $\Delta=B_4$.

Suppose now that $X \neq Q, P$. Then the group P/Q is not prime. Tables in [OV] imply that $\Delta = A_l$, where $l+1$ is not prime, or $\Delta = D_l$. If $\Delta \neq A_7, A_8, D_8$, then one can check directly that every element of P, whose length is equal to the length of a root, is a root itself. Therefore if $\Delta \neq A_7, A_8, D_8$, then $N \subset \mathrm{Aut}(\Delta)$.

The system of weights of the representation φ is invariant under $N \subset \mathrm{Aut}(\Delta)$. Thus, N coincides with the image of $N_{\widetilde{H}}(\mathfrak{t})$ in $\mathrm{GL}(\mathfrak{t}^*)$. So we are done.

Now we construct embeddings of $\mathfrak{h} = \mathfrak{sp}_8, \mathfrak{so}_9, \mathfrak{sl}_8, \mathfrak{sl}_9, \mathfrak{so}_{16}$, for which ψ_1 is not birational.

Suppose $\mathfrak{h} = \mathfrak{sp}_8$. Let $\varphi : \mathfrak{h} \to \mathfrak{gl}_{14}$ be the irreducible representation with the highest weight π_2. Let us choose the orthonormal basis $\varepsilon_1, \ldots, \varepsilon_4 \in \mathfrak{t}$, so that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, 2, 3$, $\alpha_4 = 2\varepsilon_4$. The weights of φ are $\varepsilon_i + \varepsilon_j, i \neq j$, with multiplicity 1 and 0 with multiplicity 2. The stabilizer of this weight system in $\mathrm{GL}(\mathfrak{t}^*)$ is just $\mathrm{Aut}(D_4)$. Therefore $N_G(\mathfrak{t})/Z_G(\mathfrak{t}) \cong \mathrm{Aut}(D_4)$, while $N_{\overline{H}(\mathfrak{t})}/Z_{\overline{H}}(\mathfrak{t})$ is the Weyl group of Δ and has index 3 in $N_G(\mathfrak{t})/Z_G(\mathfrak{t})$.

The algebras $\mathfrak{h} = \mathfrak{so}_9, \mathfrak{sl}_8, \mathfrak{sl}_9, \mathfrak{so}_{16}$ can be embedded into the exceptional algebras $\mathfrak{f} = F_4, E_7, E_8, E_8$, respectively, as regular subalgebras. Suppose that $\varphi : \mathfrak{h} \hookrightarrow \mathfrak{gl}_n$ is the composition of this embedding and some embedding $\rho : \mathfrak{f} \hookrightarrow \mathfrak{gl}_n$. Analogously to the case $\mathfrak{h} = \mathfrak{sp}_8$ one can show that $N_{\overline{H}}(\mathfrak{t})/Z_{\overline{H}}(\mathfrak{t})$ is not equal to $N_G(\mathfrak{t})/Z_G(\mathfrak{t})$ because the last group is the Weyl group of $\mathfrak{f}$. This completes the proof of the theorem. □

4. The algebra $\mathbb{C}[\mathfrak{h}]^H$

It is known (see [Bou], Chapter 8,§8) that for a connected semisimple group H the vector space $\mathbb{C}[\mathfrak{h}]^H$ is generated by polynomials $\mathrm{tr}(\rho(x)^n)$, where ρ runs over the set of all representations of $\mathfrak{h}$. In this section we generalize this result for groups H such that H° is algebraically simply connected (a reductive algebraic group is called algebraically simply connected if it is a direct product of a torus and a simply connected semisimple group).

Let H be a (possibly disconnected) reductive algebraic group and $\mathfrak{h}$ be its Lie algebra. Denote by $\mathfrak{R}(\mathfrak{h})$ the set of all equivalence classes of representations of $\mathfrak{h}$. Define an action of the group H on $\mathfrak{R}(\mathfrak{h})$: for $h \in H, \rho \in \mathfrak{R}(\mathfrak{h})$ we put $(h.\rho)(x) = \rho(\mathrm{Ad}(h)^{-1}x)$ for all $x \in \mathfrak{h}$. It is obvious, that $\mathfrak{R}^{H^\circ} = \mathfrak{R}$. If ρ is the differential of a representation of H, then $h.\rho = \rho$ for any $h \in H$.

Suppose $\rho \in \mathfrak{R}(\mathfrak{h})^H$. Then $\mathrm{tr}(\rho(x)^n) \in \mathbb{C}[\mathfrak{h}]^H$ for all $n \in \mathbb{N}$. Indeed, for $h \in H, x \in \mathfrak{h}$ we have $h.\mathrm{tr}(\rho(x)^n) = \mathrm{tr}(\rho(\mathrm{Ad}(h)^{-1}x)^n) = \mathrm{tr}((h.\rho)(x)^n) = \mathrm{tr}(\rho(x)^n)$.

Proposition 4.1. *Suppose that H° is an algebraically simply connected group. The vector space $\mathbb{C}[\mathfrak{h}]^H$ is generated by $\mathrm{tr}(\rho(x)^n)$, where ρ runs over the set of all representations of H.*

Proof. Suppose the group H is connected. Then $\mathbb{C}[\mathfrak{h}]^H \cong \mathbb{C}[\mathfrak{z}(\mathfrak{h})] \otimes \mathbb{C}[\mathfrak{h}']^H$, where $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$. Let ρ be an irreducible representation of $\mathfrak{h}'$, and χ be a character of the torus $Z(H)^\circ$. Put $\rho_m = \rho \otimes m\chi$. Then

$$\mathrm{tr}(\rho_m(x)^n) = \sum_{i=0}^{m} \binom{n}{i} m^{n-i} \chi(x)^{n-i} \,\mathrm{tr}(\rho(x)^i) \tag{4.1}$$

It follows from (4.1) that for all $k, l \in \mathbb{N}$ the polynomial $\chi(x)^k \,\mathrm{tr}(\rho(x)^l) \in \mathbb{C}[\mathfrak{h}]^H$ is a linear combination of polynomials $\mathrm{tr}(\rho_m(x)^{k+l})$. But the linear space generated by polynomials of the form $\chi(x)^k \,\mathrm{tr}(\rho(x)^l)$, where ρ, χ are as above, coincides with $\mathbb{C}[\mathfrak{h}]^H$. This completes the proof in the case of a connected group H.

Now we consider the general case. We have

$$\mathbb{C}[\mathfrak{h}]^H = \{ \sum_{h \in H/H^\circ} h.f \mid f \in \mathbb{C}[\mathfrak{h}]^{H^\circ} \}.$$

Therefore, the vector space $\mathbb{C}[\mathfrak{h}]^H$ is generated by elements of the form

$$\sum_{h\in H/H^\circ} h.\operatorname{tr}(\rho(x)^n), \tag{4.2}$$

where ρ is a representation of H°. Denote by $\widetilde{\rho}$ a representation of H induced from ρ. The corresponding representation $\widetilde{\rho}$ of $\mathfrak{h}$ is given by $\widetilde{\rho}=\sum_{h\in H/H^\circ} h.\rho$. The polynomial (4.2) is just $\operatorname{tr}(\widetilde{\rho}(x)^n)$. □

Let I be a set of representations of H. Denote by $\mathbb{C}[\mathfrak{h}^n]^I$ the subalgebra of $\mathbb{C}[\mathfrak{h}^n]^H$ generated by polynomials of the form $\operatorname{tr}(\rho(x)^m)$, where $\rho\in I$. When $I=\{\rho\}$, we write $\mathbb{C}[\mathfrak{h}^n]^\rho$ instead of $\mathbb{C}[\mathfrak{h}^n]^{\{\rho\}}$.

It is known from classical invariant theory (see, for example, [PV]) that if $H=\mathrm{GL}_m,\mathrm{SL}_m, \emptyset_m,\mathrm{Sp}_{2m}$, then $\mathbb{C}[\mathfrak{h}^n]^H=\mathbb{C}[\mathfrak{h}^n]^\rho$, where ρ is the tautological representation of the group H. Now let H be one of the exceptional simple groups G_2,F_4,E_6,E_7,E_8 and ρ be the simplest (=non-trivial irreducible of minimal dimension) representation of the group H. It was shown by several authors (see [DQ] for references) that $\mathbb{C}[\mathfrak{h}]^H=\mathbb{C}[\mathfrak{h}]^\rho$.

5. Linear equivalence of embeddings

Let G,H be reductive algebraic groups and $\mathfrak{g},\mathfrak{h}$ be the Lie algebras of G and H, respectively. We say that two homomorphisms $P_1,P_2:H\to G$ are *equivalent* (or, more precisely, *G-equivalent*) if there exists $g\in G$ such that $gP_1(x)g^{-1}=P_2(x)$ for all $x\in H^\circ$. Further, we say that P_1,P_2 are *linearly equivalent* (or *linearly G-equivalent*) if for every representation $P:G\to\mathrm{GL}(V)$ representations $P\circ P_1,P\circ P_2$ are $\mathrm{GL}(V)$-equivalent. It is obvious, that equivalent homomorphisms are linearly equivalent. Homomorphisms $\rho_1,\rho_2:\mathfrak{h}\to\mathfrak{g}$ are said to be G-equivalent (resp., linearly G-equivalent) if there exist equivalent (resp., linearly equivalent) homomorphisms $P_1,P_2:H\to G$ with $dP_1=\rho_1, dP_2=\rho_2$.

Let $P:H\to G$ be a homomorphism, $\rho:\mathfrak{h}\to\mathfrak{g}$ its tangent homomorphism. Denote by ψ_n^ρ the natural morphism $\mathfrak{h}^n//H\to\rho(\mathfrak{h})^n//G$.

The set $\rho(\mathfrak{h})^n//G$ can be identified with the set of equivalence classes $(\rho(x_1),\dots,\rho(x_n))$, where $x_1,\dots,x_n$ generate a reductive subalgebra of $\mathfrak{h}$, modulo G-conjugacy. Therefore Lemma 2.3 implies that $\psi_n^\rho, n>1$, is bijective iff for every reductive Lie algebra $\mathfrak{f}$ and embeddings $\rho_1,\rho_2:\mathfrak{f}\to\mathfrak{h}$ the following condition is fulfilled:

if $\rho\circ\rho_1,\rho\circ\rho_2$ are G-equivalent, then ρ_1,ρ_2 are H-equivalent.

Similarly, we get the following

Proposition 5.1. *Let $H,G,\mathfrak{h},\mathfrak{g},\rho$ be as above. The following conditions are equivalent:*

(i) *The map ψ_1^ρ is injective.*
(ii) *For every pair (x_1,x_2) of semisimple elements of $\mathfrak{h}$ if $\rho(x_1)\sim_G\rho(x_2)$, then $x_1\sim_H x_2$.*
(iii) *For any diagonalizable Lie algebra $\mathfrak{t}$ and embeddings $\rho_1,\rho_2:\mathfrak{t}\hookrightarrow\mathfrak{h}$ if ρ_1,ρ_2 are H-equivalent, then $\rho\circ\rho_1,\rho\circ\rho_2$ are G-equivalent.*

The following proposition is a generalization of Theorem 1.1. from [Dy]

Proposition 5.2. *Let $G,\mathfrak{h},\mathfrak{g}$ be as above, $\mathfrak{t}\subset\mathfrak{h}$ be a Cartan subalgebra, $P_1,P_2:H\to G$ be some homomorphisms, $\rho_1=dP_1,\rho_2=dP_2$. The following conditions are equivalent*

(i) *ρ_1 and ρ_2 are linearly G-equivalent.*
(ii) *$\rho_1|_{\mathfrak{t}}$ and $\rho_2|_{\mathfrak{t}}$ are G-equivalent.*

Proof. *(ii)* $\Rightarrow$ *(i)*. Replacing ρ_1 by $\mathrm{Ad}(g)\circ\rho_1, g\in G$, if necessary, we may assume that $\rho_1|_{\mathfrak{t}}=\rho_2|_{\mathfrak{t}}$. The required result follows from the fact that a representation of $\mathfrak{h}$ is uniquely determined by the collection of its weights and their multiplicities.

(i) $\Rightarrow$ *(ii)*. We may assume that $\mathfrak{h}=\mathfrak{t}$. By Proposition 5.1 it is enough to prove the following claim:

Let x_1, x_2 be semisimple elements of $\mathfrak{g}$ such that for every representation $P : G \to \mathrm{GL}(V)$ the matrices $\rho(x_1), \rho(x_2)$ are conjugate, where $\rho = dP$. Then x_1, x_2 are conjugate (with respect to the adjoint action of G).

First we suppose that G° is algebraically simply connected. We see that $\mathrm{tr}(\rho(x_1)^n) = \mathrm{tr}(\rho(x_2)^n)$ for all $P : G \to \mathrm{GL}(V)$. It follows from Proposition 4.1 that $f(x_1) = f(x_2)$ for any $f \in \mathbb{C}[\mathfrak{g}]^G$. Since x_1, x_2 are semisimple, we have $x_1 \sim_G x_2$.

Our claim in the general case is now a consequence of the following

Lemma 5.3. *For every reductive group G there exist a group $\widetilde{G}$ and a surjective covering $\pi : \widetilde{G} \to G$ such that $\mathrm{Ad}(\widetilde{G}) = \mathrm{Ad}(G)$ and $\widetilde{G}^\circ$ is algebraically simply connected.*

Proof of Lemma. There exist a finite subgroup $F \subset G$ such that $G = FG^\circ$ (see [Vi], Proposition 7). Let $\widetilde{G'}$ be the simply connected covering of (G, G). The group $\widetilde{G} = F \leftthreetimes (Z(G^\circ) \times \widetilde{G'})$ and the natural morphism $\pi : \widetilde{G} \to G$ has the required properties. □

□

Remark 5.4. Let I be a set of representations of G such that the polynomials $\mathrm{tr}(dP(x)^n), P \in I$, generate the algebra $\mathbb{C}[\mathfrak{g}]^G$. It follows from the previous proof that $\rho_1|_\mathfrak{t}, \rho_2|_\mathfrak{t}$ are H-equivalent iff the representations $dP \circ \rho_1, dP \circ \rho_2$ are $\mathrm{GL}(V)$-equivalent for any representation $P \in I$.

6. Bijectivity of $\psi_2 : \mathfrak{h}^2//\overline{H} \to \mathfrak{h}^2//G$

Theorem 6.1. *Let $\mathfrak{h}$ be a simple Lie algebra, H an algebraic group with Lie algebra $\mathfrak{h}$. The following conditions are equivalent:*

(i) $\mathfrak{h} = \mathfrak{sl}_n, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}, G_2, F_4$ *or* $\mathfrak{h} = \mathfrak{so}_{2n}, n > 3$, *and the group* $\mathrm{Ad}(H)$ *contains an involutory outer automorphism of the algebra* $\mathfrak{h}$.

(ii) *For any reductive algebraic Lie algebra $\mathfrak{f}$ and any pair of linearly H-equivalnent homomorhisms $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ ρ_1 and ρ_2 are H-equivalent.*

Theorem 6.1 is proved in Sections 7-10.

Proof of Theorem 1.3. It is enough to prove that for every reductive Lie algebra $\mathfrak{f}$ and every embeddings $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ if ρ_1, ρ_2 are G-equivalent then they are H-equivalent.

It follows from the bijectivity of ψ_1 that $\rho_1|_\mathfrak{t}, \rho_2|_\mathfrak{t}$ are H-equivalent. It is enough to prove that the latter implies ρ_1, ρ_2 are H-equivalent. One may assume that $\rho_1|_\mathfrak{t} = \rho_2|_\mathfrak{t}$. Let $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}_1 \oplus \ldots\ldots \oplus \mathfrak{h}_k$, where $\mathfrak{h}_i$ is a simple noncommutative ideal. Now it's enough to consider the case $\overline{H} = H_1 \times \ldots \times H_k$, where $H_i = \varnothing_{2m}$ if $\mathfrak{h}_i = \mathfrak{so}_{2m}, m > 3$, $H_i = \mathrm{Int}(\mathfrak{h}_i)$ otherwise. Denote by π_i the projection from $\mathfrak{h}$ to $\mathfrak{h}_i$. It's enough to prove that there is $h_k \in H_k$ such that $\mathrm{Ad}(h_k) \circ \pi_k \circ \rho_1 = \pi_k \circ \rho_2$. By Proposition 5.2, homomorphisms $\pi_k \circ \rho_1, \pi_k \circ \rho_2$ are linearly H-equivalent. It remains to use Theorem 6.1. □

Proposition 6.2. *Let G be a reductive algebraic group such that G° is simply connected, $\mathfrak{g}$ the Lie algebra of G. There exists a representation $P : G \to \mathrm{GL}(V)$ such that $\rho = dP$ is an embedding and $\psi_1^\rho : \mathfrak{g}//G \to \rho(\mathfrak{g})//\,\mathrm{GL}(V)$ is injective.*

Proof. We need the following lemma

Lemma 6.3. *Let $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(U_1), \rho_2 : \mathfrak{g} \to \mathfrak{gl}(U_2)$ be representations, $n = \dim U_1 + 1$. Put $U = U_1 \oplus nU_2$, $\rho = \rho_1 + n\rho_2$. For any two semisimple elements $x, y \in \mathfrak{g}$ if the matrices $\rho(x), \rho(y)$ are similar, then for $i = 1, 2$ the matrices $\rho_i(x), \rho_i(y)$ are similar.*

Proof of Lemma 6.3. Denote by $\lambda_1, \ldots, \lambda_k$ the different eigenvalues of matrices $\rho(x), \rho(y)$ and by $m_1, \ldots, m_k$ their multiplicities. Both $\rho_1(x)$ and $\rho_1(y)$ (respectively, $\rho_2(x), \rho_2(y)$) have eigenvalues $\lambda_1, \ldots, \lambda_k$ with multiplicities $n\{\frac{m_1}{n}\}, \ldots, n\{\frac{m_k}{n}\}$ (respectively, $[\frac{m_1}{n}], \ldots, [\frac{m_k}{n}]$). Therefore, $\rho_i(x), \rho_i(y)$ are similar for $i = 1, 2$. □

It follows from Proposition 4.1 that there exist representations $P_i : G \to \mathrm{GL}(U_i), i = \overline{1, m}$, such that the algebra $\mathbb{C}[\mathfrak{g}]^G$ is generated by polynomials of the form $\mathrm{tr}(dP_i(x)^n)$.

For $i = 1, \dots, m$ we define a positive integer n_i, a vector space $\widetilde{U}_i$ and a representation $\widetilde{P}_i : G \to \mathrm{GL}(\widetilde{U}_i)$ by formulas

$$n_1 = 1, \widetilde{U}_1 = U_1, \widetilde{P}_1 = P_1.$$

$$n_i = \dim \widetilde{U}_{i-1} + 1, \widetilde{U}_i = \widetilde{U}_{i-1} \oplus n_i U_i, \widetilde{P}_i = \widetilde{P}_{i-1} + n_i P_i.$$

By Lemma 6.3, for any semisimple $x, y \in \mathfrak{g}$ if the matrices $d\widetilde{P}_m(x), d\widetilde{P}_m(y)$ are similar, then for $i = \overline{1, m}$ the matrices $dP_i(x), dP_i(y)$ are similar. Thus $f(x) = f(y)$ for all $f \in \mathbb{C}[\mathfrak{g}]^G$ and so $x \sim_G y$. It follows from Proposition 5.1 that $\psi_1^{dP_m}$ is bijective. □

Proof of Proposition 1.4. First, suppose that $G = \mathrm{GL}_m$. Let $\mathfrak{f}$ be a reductive Lie algebra, $\rho_1, \rho_2 : \mathfrak{f} \hookrightarrow \mathfrak{h}$ linearly H-equivalent but not equivalent embeddings. Such ρ_1, ρ_2 exist by Theorem 6.1. If ψ_2 is bijective, then ρ_1, ρ_2 are H-equivalent. Contradiction.

Now we consider the general case. Since $\mathfrak{h} \subset [\mathfrak{g}, \mathfrak{g}]$, one may assume that G is a semisimple group. Let G_1 be a simply connected group with Lie algebra $\mathfrak{g}$. Changing G with $G_1 \rightthreetimes (\mathrm{Ad}(G)/\,\mathrm{Int}(\mathfrak{g}))$, we may assume that G° is simply connected. By Proposition 6.2, there exists a homomorphism $P : G \to \mathrm{GL}(V)$ such that $\rho = dP$ is an embedding and ψ_1^ρ is bijective. Since ψ_1^ρ is injective, it follows that $\overline{G} := N_{\mathrm{GL}(V)}(\rho(\mathfrak{g}))/Z_{\mathrm{GL}(V)}(\rho(\mathfrak{g}))$ is a group of type I. Therefore, by Theorem 1.3, $\psi_2^\rho : \mathfrak{g}^2/\!/G \to \mathfrak{g}^2/\!/\,\mathrm{GL}(V)$ is bijective. Denote by $\widetilde{\rho}$ the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Since ψ_2^ρ is bijective, the group $N_{\mathrm{GL}(V)}(\rho(\mathfrak{h}))/Z_{\mathrm{GL}(V)}(\rho(\mathfrak{h})) \subset \mathrm{Aut}(\mathfrak{h})$ coincides with $\overline{H}$. Now it follows from the first part of the proof that the map $\psi_2^{\rho\circ\widetilde{\rho}}$ is not bijective. But $\psi_2^{\rho\circ\widetilde{\rho}} = \psi_2^\rho \circ \psi_2$. Therefore ψ_2 is not bijective. □

Proposition 6.4. *Let $G, \mathfrak{g}, H, \mathfrak{h}$ be such as in Proposition 5.1, $P : H \to G$ a homomorphism, $\rho = dP$, $\mathfrak{f}$ a reductive Lie algebra, and $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ embeddings. Suppose that the equivalent conditions of Proposition 5.1 are fulfilled and ρ_1, ρ_2 are G-equivalent. If $\mathfrak{f} = \mathfrak{sl}_2$ or $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f})$ are regular subalgebras of $\mathfrak{h}$, then ρ_1 and ρ_2 are H-equivalent.*

Proof. Suppose $\mathfrak{f} = \mathfrak{sl}_2$. Let (e, h, f) be a standard basis of $\mathfrak{f}$, i.e. $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. Since $\rho_1(h) \sim_G \rho_2(h)$ we see that $\rho_1(h) \sim_H \rho_2(h)$. We may assume that $\rho_1(h) = \rho_2(h)$. There exists $g \in Z_H(\rho_1(h))$ such that $\mathrm{Ad}(g) \circ \rho_1 = \rho_2$ (see, for example, [Bou], ch.8, §11).

Now suppose that $\rho_1(\mathfrak{f})$ and $\rho_2(\mathfrak{f})$ are regular subalgebras of $\mathfrak{h}$. Denote by $\mathfrak{s}, \mathfrak{t}$ Cartan subalgebras of $\mathfrak{f}, \mathfrak{h}$, respectively. There exists $h \in H$ such that $\mathrm{Ad}(h)\rho_1|_{\mathfrak{s}} = \rho_2|_{\mathfrak{s}}$. Therefore the proof reduces to the case when $\rho_1|_{\mathfrak{s}} = \rho_2|_{\mathfrak{s}}$. Since all Cartan subalgebras of $\mathfrak{z}_{\mathfrak{g}}(\rho_1(\mathfrak{s}))$ are $Z_G(\rho_1(\mathfrak{s}))$-conjugate, one also may assume that $\mathfrak{t}$ normalizes $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f})$. Let $\alpha \in \Delta(\mathfrak{f})$ and $e_\alpha \in \mathfrak{f}^\alpha$ be nonzero. Since $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f})$ are regular subalgebras, $\rho^i(e_\alpha) \subset \mathfrak{h}^{\beta_i}, i = 1, 2$, for some roots β_1, β_2 of $\mathfrak{h}$. This implies $\rho_i(\alpha^\vee) \subset \mathbb{C}\beta_i^\vee$. Since $\rho^1|_{\mathfrak{s}} = \rho^2|_{\mathfrak{s}}$, it follows that $\rho_1(e_\alpha) = c_\alpha \rho_2(e_\alpha)$ for some $c_\alpha \in \mathbb{C}$. There exists $t \in \rho_1(\mathfrak{s})$ such that $\exp(\mathrm{ad}\, t) \circ \rho_1 = \rho_2$, see [Bou], ch.8,§5. □

7. Cases $\mathfrak{h} = \mathfrak{sl}_n, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}$

We show that linearly H-equivalent reductive embeddings are H-equivalent for every group H with Lie algebra $\mathfrak{h}$.

Let $\mathfrak{f}$ be a reductive Lie algebra and $\rho_1, \rho_2 : \mathfrak{f} \hookrightarrow \mathfrak{h}$ be linearly H-equivalent embeddings. We have to prove that ρ_1, ρ_2 are H-equivalent. By Proposition 5.2, we may assume that $\rho_1|_{\mathfrak{t}} = \rho_2|_{\mathfrak{t}}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{f}$. Now it is enough to show that ρ_1, ρ_2 are H°-equivalent. Denote by ρ the tautological representation of $\mathfrak{h}$. Recall that $\mathbb{C}[\mathfrak{h}^2]^\rho = \mathbb{C}[\mathfrak{h}^2]^{H^\circ}$. It follows that the embeddings ρ_1 and ρ_2 are H°-equivalent.

8. Case $\mathfrak{h}=\mathfrak{so}_{2n}, n>3$

First suppose that $\mathrm{Ad}(H)$ contains an involutory outer automorphism of $\mathfrak{h}$. Then linearly H-equivalent reductive embeddings are H-equivalent. One can prove this analogously to the previous section using the fact that $\mathbb{C}[\mathfrak{h}^n]^\rho=\mathbb{C}[\mathfrak{h}^n]^H$, where ρ is the tautological representation of $\mathfrak{h}$, $H=\mathrm{O}_{2n}$.

Now suppose that $\mathrm{Ad}(H)=\mathrm{Int}(\mathfrak{h})$. Denote by τ the tautological representation of $\mathfrak{h}$ and by θ any outer involutory automorphism of $\mathfrak{h}$. For the proof of the next proposition see [Dy].

Proposition 8.1. *Let $\mathfrak{f}$ be a reductive Lie algebra and $\rho:\mathfrak{f}\to\mathfrak{h}$ be an embedding. Suppose that*

1. *The representation $\tau\circ\rho:\mathfrak{f}\to\mathfrak{gl}_{2n}$ has zero weight.*
2. *All irreducible components of $\tau\circ\rho$ have even dimension.*

Then the embeddings $\rho,\theta\circ\rho$ are linearly H-equivalent but not equivalent.

Now it is enough to construct an embedding $\rho:\mathfrak{f}\hookrightarrow\mathfrak{h}$ (or representation $\tau\circ\rho$) satisfying the both conditions of Proposition 8.1. Denote by ρ_1 the adjoint representation of $\mathfrak{sl}_3$ (of dimension 8), by ρ_2 the exterior square of the tautological representation of $\mathfrak{so}_5$ (of dimension 10) and by ρ_0 the tautological representation of $\mathfrak{so}_4$. For $n=2k$ we put $\mathfrak{f}=\mathfrak{sl}_3\oplus\mathfrak{so}_4^{k-4}$, $\tau\circ\rho=\rho_1\oplus(k-4)\rho_0$. For $n=2k+1$ put $\mathfrak{f}=\mathfrak{so}_5\oplus\mathfrak{so}_4^{k-4},\tau\circ\rho=\rho_2\oplus(k-4)\rho_0$.

It remains to consider the case $\mathfrak{h}=\mathfrak{so}_8, |\mathrm{Ad}(H)/\mathrm{Int}(\mathfrak{h})|=3$. It is enough to prove that $\rho_1,\theta\circ\rho_1:\mathfrak{sl}_3\hookrightarrow\mathfrak{so}_8$ are not equivalent.

Assume the converse. There exists $h\in H$ such that $(\mathrm{Ad}(h)\theta)\circ\rho_1=\rho_1$. The order of the image of $\mathrm{Ad}(h)\theta$ in the group $\mathrm{Aut}(\mathfrak{h})/\mathrm{Int}(\mathfrak{h})$ is 2. This contradicts Proposition 8.1.

9. Case $\mathfrak{h}=E_l, l=6,7,8$

There exists a Levi subalgebra $l\subset\mathfrak{h}$ isomorphic to $\mathfrak{so}_{10}\times\mathbb{C}^{l-5}$. Put $\mathfrak{f}=\mathfrak{so}_5\times\mathbb{C}^{l-5}$. Denote by ρ^1,ρ^2 embeddings of $\mathfrak{f}$ into l satisfying the following conditions

1. $\rho^1|_{\mathfrak{z}(\mathfrak{f})}=\rho^2|_{\mathfrak{z}(\mathfrak{f})}$ is an isomorphism of $\mathfrak{z}(\mathfrak{f})$ and $\mathfrak{z}(l)$.
2. $\rho^1|_{\mathfrak{so}_5}=\rho_2,\rho^2|_{\mathfrak{so}_5}=\theta\circ\rho_2$, where ρ_2 is the exterior square of the tautological representation of $\mathfrak{so}_5$, θ is an involutory outer automorphism of $\mathfrak{so}_{10}$.

Since $\rho_2,\theta\circ\rho_2:\mathfrak{so}_5\hookrightarrow\mathfrak{so}_{10}$ are linearly SO_{10}-equivalent, we see that ρ^1,ρ^2 are linearly $\mathrm{Int}(\mathfrak{h})$-equivalent.

Assume that there exists $h\in\mathrm{Ad}(H)$ such that $h\circ\rho^1=\rho^2$. Denote by L' a connected subgroup of H with Lie algebra $[l,l]$. It is well known that the centralizer of an algebraic subtorus in a connected reductive algebraic group is connected. By Proposition 8.1, $h|_{[l,l]}\notin\mathrm{Ad}(L')$. Since $h\in Z_H(\mathfrak{z}(l))$, we have $h\notin\mathrm{Int}(\mathfrak{h})$. Thus, $\mathfrak{h}=E_6,\mathrm{Ad}(H)=\mathrm{Aut}(\mathfrak{h})$.

Denote by $\mathfrak{t}$ a Cartan subalgebra of l. One may assume that $\mathfrak{t}\cap[l,l]$ is θ-invariant and that θ acts on $\mathfrak{t}\cap[l,l]$ by a reflection. The centralizer of $\rho_2(\mathfrak{so}_5)$ in SO_{10} coincides with the center of SO_{10}. Hence $\mathrm{Ad}(h)|_{[l,l]}=\theta$ and $\mathrm{Ad}(h)|_{\mathfrak{t}}$ is a reflection because $\mathrm{Ad}(h)$ acts trivially on $\mathfrak{z}(l)$. The subgroup of $N_H(\mathfrak{t})$ generated by all reflections is the Weyl group of $\mathfrak{h}$. Therefore $h\in\mathrm{Int}(\mathfrak{h})$. Contradiction.

10. Cases $\mathfrak{h}=G_2,F_4$

Since the algebra $\mathfrak{h}$ has no outer automorphisms, one may assume that H is connected.

If $\mathfrak{h}=G_2$ the statement of Theorem 6.1 follows from Proposition 6.4. In the sequel we consider the case $\mathfrak{h}=F_4$.

Let $\mathfrak{f}$ be a reductive Lie algebra, $\rho_1,\rho_2:\mathfrak{f}\hookrightarrow\mathfrak{h}$ linearly H-equivalent embeddings. We have to prove that ρ_1,ρ_2 are equivalent.

Assume the converse. It follows from Proposition 6.4 that $\mathrm{rank}\,\mathfrak{f}<4$.

Lemma 10.1. *Let H be a reductive algebraic group, $\mathfrak{f}$ a reductive Lie algebra such that $\mathfrak{f} \cong \mathfrak{s} \oplus \mathfrak{f}_1$, where $\mathfrak{s}, \mathfrak{f}_1$ are ideals of $\mathfrak{f}$ and* $\operatorname{rank}\mathfrak{s} = 1$. *Suppose $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ are linearly H-equivalent embeddings. Then*

(1) *There exists $h \in H$ such that $\operatorname{Ad}(h) \circ \rho_1$ and ρ_2 coincide on $\mathfrak{s}$.*
(2) *Suppose that ρ_1, ρ_2 coincide on $\mathfrak{s}$. Then $\rho_1|_{\mathfrak{f}_1}, \rho_2|_{\mathfrak{f}_1}$ are linearly $Z_H(\rho_1(\mathfrak{s}))$-equivalent.*
(3) *Under assumptions of (2) ρ_1, ρ_2 are H-equivalent iff $\rho_1|_{\mathfrak{f}_1}, \rho_2|_{\mathfrak{f}_1}$ are $Z_H(\rho_1(\mathfrak{s}))$-equivalent.*

Proof. The third assertion is obvious. Propositions 5.2 and 6.4 imply assertion (1),(2) for $\mathfrak{s} \cong \mathbb{C}$ and assertion (1) for $\mathfrak{s} = \mathfrak{sl}_2$, respectively.

Now one may assume that $\mathfrak{s} \cong \mathfrak{sl}_2$ and $\rho_1|_{\mathfrak{s}} = \rho_2|_{\mathfrak{s}}$. Denote by x a general semisimple element of $\mathfrak{f}_1$. By Proposition 5.2, it is enough to show that there exists $g \in Z_H(\rho_1(\mathfrak{s}))$ such that $\operatorname{Ad}(g)\rho_1(x) = \rho_2(x)$. Denote by e, h, f a standard basis of $\rho_1(\mathfrak{s})$. Since ρ_1, ρ_2 are linearly H-equivalent, there exist $g_1 \in Z_H(h)$ such that $\operatorname{Ad}(g_1)\rho_1(x) = \rho_2(x)$. Analogously to the proof of Proposition 6.4, there exists $g_2 \in Z_H(\rho_2(x))$ such that $\operatorname{Ad}(g_2)\operatorname{Ad}(g_1)\rho_1, \rho_2$ coincide on $\mathfrak{s}$. The element $g = g_2 g_1$ has the required properties. $\square$

Suppose that $\mathfrak{f}$ is not simple. Then $\mathfrak{f}$ satisfies conditions of Lemma 10.1. It is easy to see that $Z_H(\rho_1(\mathfrak{s}))$ is a group of type I. Rank of semisimple part of $Z_H(\rho_1(\mathfrak{s}))$ is less than 4. It follows from results of Sections 8,9, that if ρ_1, ρ_2 are linearly equivalent, then they are equivalent (see the proof of Theorem 1.3) . So, we may assume that $\mathfrak{f}$ is simple. Proposition 6.4 implies that $\operatorname{rank}\mathfrak{f} > 1$.

Now we prove that the subalgebras $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f}) \subset \mathfrak{h}$ are not regular. Assume the converse. To be definite, let $\rho_1(\mathfrak{f})$ be a regular subalgebra of $\mathfrak{h}$. By Proposition 6.4, $\rho_2(\mathfrak{f})$ is not regular. Since the representations $\operatorname{ad}\circ\rho_1, \operatorname{ad}\circ\rho_2$ are equivalent, we obtain that $\dim \mathfrak{z}_{\mathfrak{h}}(\rho_2(\mathfrak{f})) = \dim \mathfrak{z}_{\mathfrak{h}}(\rho_1(\mathfrak{f})) \geqslant 4 - \operatorname{rank}\mathfrak{f}$. Since for any reductive Lie algebra $\mathfrak{s}$ the number $\dim\mathfrak{s} - \operatorname{rank}\mathfrak{s}$ is even, $\operatorname{rank}\mathfrak{z}_{\mathfrak{h}}(\rho_2(\mathfrak{f})) \leqslant \operatorname{rank}\mathfrak{z}_{\mathfrak{h}}(\rho_1(\mathfrak{f})) - 2$. Therefore, $\operatorname{rank}\mathfrak{z}_{\mathfrak{h}}(\rho_2(\mathfrak{f})) = 0$. Contradiction.

It follows from [Dy] that every simple subalgebra in $\mathfrak{h}$ of rank greater then 1 is contained in a maximal regular subalgebra. There are three maximal regular subalgebras $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \subset \mathfrak{h}$, $\mathfrak{h}_1 \cong \mathfrak{so}_9, \mathfrak{h}_2 \cong \mathfrak{sp}_6 \times \mathfrak{sl}_2, \mathfrak{h}_3 \cong \mathfrak{sl}_3 \times \mathfrak{sl}_3$. See, for example, [Dy] (there is a mistake in this paper: the subalgebra of F_4 isomorphic to $\mathfrak{sl}_4 \times \mathfrak{sl}_2$ is not maximal, it is contained in $\mathfrak{h}_1$). One may assume that $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ contain a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h}$.

All simple subalgebras of rank greater then 1 in $\mathfrak{sp}_6$ are regular. All embeddings $\rho : \mathfrak{f} \hookrightarrow \mathfrak{h}_1$ such that the subalgebra $\rho(\mathfrak{f})$ is not regular are listed up to $\operatorname{Int}(\mathfrak{h}_1)$-conjugacy in the following table:

$\mathfrak{f}$	ρ
$\mathfrak{sl}_4$	$R(\pi_2) \oplus 3R(0)$
$\mathfrak{so}_7$	$R(\pi_3) \oplus R(0)$
G_2	$R(\pi_1) \oplus 2R(0)$
$\mathfrak{so}_5$	$R(\pi_2) \oplus R(\pi_2) \oplus R(0)$
$\mathfrak{sl}_3$	$\operatorname{ad} \oplus R(0)$

In the second column linear representations in $\mathbb{C}^9$ corresponding to ρ are listed. They determine ρ up to $\operatorname{Int}(\mathfrak{h}_1)$-conjugacy.

Since all simple subalgebras of rank greater then 1 in $\mathfrak{h}_3$ are isomorphic to $\mathfrak{sl}_3$, we see that $\mathfrak{f} \cong \mathfrak{sl}_3$. Denote by ρ_1 the embedding of $\mathfrak{sl}_3$ into $\mathfrak{so}_9$ listed in the table. It follows from tables in [Dy] that the restriction of simplest representation of $\mathfrak{h}$ into $\rho_1(\mathfrak{f})$ is isomorphic to $3\operatorname{ad} \oplus 2R(0)$.

It is easy to see that $N_H(\mathfrak{h}_3)/N_H(\mathfrak{h}_3)^\circ$ is a group of order 2. The group $N_H(\mathfrak{h}_3)$ contains an outer automorphism of $\mathfrak{h}_3$, which acts on $\mathfrak{t}$ by multiplication by -1.

Now we introduce some notation. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{h}$, $\varepsilon_i, i=\overline{1,4}$, its orthonormal basis, so that

$$\Delta(\mathfrak{h})=\{\pm\varepsilon_i\pm\varepsilon_j, i\neq j, \pm\varepsilon_i, \frac{\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}\}.$$

Put $\alpha_1=(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)/2, \alpha_2=\varepsilon_4, \alpha_3=\varepsilon_3-\varepsilon_4, \alpha_4=\varepsilon_2-\varepsilon_3$. This is a set of simple roots of $\mathfrak{h}$. Denote by h_1,h_2 simple coroots of $\mathfrak{f}$.

A set of simple roots for $\mathfrak{h}_3$ is $(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2, \alpha_1,\alpha_3,\alpha_4$. Highest weights for the restriction of the simplest representation of $\mathfrak{h}$ into $\mathfrak{h}_3$ are $\varepsilon_1, (\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)/2, (\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)/2$.

There are two equivalence classes of the embeddings of $\mathfrak{sl}_3$ into $\mathfrak{h}_3$ up to $N_H(\mathfrak{h}_3)$-conjugacy. Only one of these embeddings is GL(26)-equivalent to ρ_1, namely the one with $h_1\mapsto\varepsilon_1+2\varepsilon_2+\varepsilon_4,\ h_2\mapsto\varepsilon_1-\varepsilon_2-2\varepsilon_4$. We denote this embedding by ρ_2.

It remains to prove that $\rho_1,\rho_2:\mathfrak{f}\hookrightarrow\mathfrak{h}$ are H-equivalent. Since an outer automorphism of $\mathfrak{f}$ is contained in both $N_H(\rho_1(\mathfrak{f})), N_H(\rho_2(\mathfrak{f}))$, it is enough to show that the subalgebras $\rho_1(\mathfrak{f}),\rho_2(\mathfrak{f})$ are H-conjugate.

Denote by $\mathfrak{h}_4$ the regular subalgebra of $\mathfrak{h}$ corresponding to the set of roots of maximal length. $\mathfrak{h}_4$ is isomorphic to $\mathfrak{so}_8$ and is contained in $\mathfrak{h}_1$. One may assume that $\rho_1(\mathfrak{f})\subset\mathfrak{h}_4$. It is clear that $N_H(\mathfrak{t})\subset N_H(\mathfrak{h}_4)$. Therefore $\mathrm{Ad}(N_H(\mathfrak{h}_4))=\mathrm{Aut}(\mathfrak{h}_4)$. It follows from description of automorphisms of finite order of simple Lie algebras (see, for example, [OV], ch. 4, §4) that there exists $t\in N_H(\mathfrak{h}_4)$ such that $\mathrm{Ad}_{\mathfrak{h}_4} t$ is an element of order 3 and $\mathfrak{h}_4^t=\rho_1(\mathfrak{f})$. $Z_H(\mathfrak{h}_4)$ is a finite group of order 2. Since $t^3\in Z_H(\mathfrak{h}_4)$, it follows that $\mathrm{Ad}(t)$ has finite order (3 or 6). $\mathfrak{h}^t$ is a regular subalgebra of rank 4 in $\mathfrak{h}$ and contains $\rho_1(\mathfrak{h})$. Thus $\mathfrak{h}^t$ is not contained in a subalgebra conjugate to $\mathfrak{h}_2$.

Let us note that every proper subalgebra of $\mathfrak{h}_1$ containing $\rho_1(\mathfrak{f})$ coincides with $\mathfrak{h}_4$ or $\rho_1(\mathfrak{f})$. Indeed, let $\widetilde{\mathfrak{f}}$ be such a subalgebra. Since centralizer of $\mathfrak{f}$ in $\mathfrak{so}_9$ is trivial, we obtain that $\widetilde{\mathfrak{f}}$ is semisimple. The algebra $\mathfrak{so}_9$ does not contain a subalgebra isomorphic to $\mathfrak{sl}_3\times\mathfrak{sl}_3$. Hence, $\widetilde{\mathfrak{f}}$ is simple. If $\widetilde{\mathfrak{f}}$ is not conjugate to $\mathfrak{so}_8$, then $\widetilde{\mathfrak{f}}$ is not regular. It follows from the previous table that $\widetilde{\mathfrak{f}}\cong\mathfrak{so}_7$ or $\mathfrak{f}$. A subalgebra of $\mathfrak{so}_7$ isomorphic to $\mathfrak{sl}_3$ is unique up to conjugation and has non-trivial centralizer. It follows that $\widetilde{\mathfrak{f}}=\mathfrak{f}$.

Since order of t is divisible by 3, $\mathfrak{h}^t$ is not conjugate to $\mathfrak{h}_4$. Thus, $\mathfrak{h}^t$ is not contained in a subalgebra conjugate to $\mathfrak{h}_1$.

It follows that $\rho_1(\mathfrak{f})$ is contained in a subalgebra conjugate to $\mathfrak{h}_3$. This completes the proof.

11. The algebra $\mathbb{C}[\mathfrak{h}^n]^{\mathrm{GL}_m}$

Proof of Proposition 1.5. It is known from classical invariant theory that the algebra $\mathbb{C}[\mathfrak{h}^n]^G$ is generated by polynomials $\mathrm{tr}(X_{i_1}X_{i_2}\ldots X_{i_k})$. Denote by A a subalgebra of the tensor algebra $T\mathfrak{h}^*$ generated by elements of the form

$$g(L_1,\ldots,L_d), \tag{11.1}$$

where $g\in S^d(\mathfrak{h}^*)^G$, L_i are Lie polynomials in $X_1,\ldots,X_k$.

Put $f_k=\mathrm{tr}(X_1\ldots X_k)$. Using polarization we reduce the required statement to the following one:

$f_k\in A$ for every positive integer k.

The proof of the last statement is by induction on k. The case $k=1$ is trivial. Now assume that we are done for $k<l$.

The symmetric group S_l acts on the space $(\mathfrak{h}^*)^{\otimes l}$ by permuting factors in tensor product. It is clear that $A\cap(\mathfrak{h}^*)^{\otimes l}$ is invariant under this action. For every transposition $\sigma_i=(ii+1)$ one has

$$(\sigma_i f_l)(X_1,\ldots,X_l)-f_l(X_1,\ldots,X_l)=\mathrm{tr}(X_1\ldots[X_i,X_{i+1}]\ldots X_k).$$

Therefore, $\sigma_i f_l - f_l \in A$. It follows that $f_l - \sigma f_l \in A$ for every $\sigma \in S_l$. Hence, $f_l \in A$ iff

$$\frac{1}{l!}\sum_{\sigma\in S_l} \sigma f_l \in A.$$

But the latter is an element of $S^l(\mathfrak{h}^*)^G$ and lies in A by definition. $\square$

Corollary 11.1. *Let $G = \mathrm{GL}_m$, $\mathfrak{h}$ be a subalgebra of $\mathfrak{g} = \mathfrak{gl}_m$, $\overline{H} = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$. Suppose that $\psi_1 : \mathbb{C}[\mathfrak{h}]^G \to \mathbb{C}[\mathfrak{h}]^{\overline{H}}$ is an isomorphism and $(\mathfrak{h}, \overline{H})$ is one of the following pairs: $(\mathfrak{sl}_k, \mathrm{Ad}(\mathrm{SL}_k))$, $(\mathfrak{so}_k, \mathrm{Ad}(\varnothing_k))$, $(\mathfrak{sp}_{2k}, \mathrm{Ad}(\mathrm{Sp}_{2k}))$. Then ψ_n is an isomorphism.*

Proof. Let ρ be a representation $\mathfrak{h} \hookrightarrow \mathfrak{g}$ and ι be the tautological representation of $\mathfrak{h}$. Proposition 1.5 implies $\mathbb{C}[\mathfrak{h}^n]^\rho = \mathbb{C}[\mathfrak{h}^n]^\iota$. It follows from classical invariant theory that $\mathbb{C}[\mathfrak{h}^n]^\iota = \mathbb{C}[\mathfrak{h}^n]^{\overline{H}}$ and we are done. $\square$

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